Last Time: Determinants

Prop: Every matrix M can be expressed as

$$M = E_n E_{n-1} \cdots E_1 \cdot \mathrm{RREF}(M)$$ ☆

Recall: det is multiplicative.

i.e. $\det(AB) = \det(A)\det(B)$.

Point: ① "Computing $\mathrm{RREF}(M)$ can also compute $\det(M)$."

② $\det(M) = \det(E_n)\det(E_{n-1}) \cdots \det(E_1) \cdot \underbrace{\det(\mathrm{RREF}(M))}_{=1 \text{ or } 0}$

Change of Basis

"with respect to"

Recall: Given basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ of v.s. V, every vector of V has a representation w.r.t. $\mathcal{B}$.

$v \in V$ can be expressed uniquely as $v = \sum_{i=1}^{n} c_i b_i$.

The corresponding representation is $[v]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$.

NB: $\mathrm{Rep}_{\mathcal{B}}(v)$ is the textbook's notation for $[v]_{\mathcal{B}}$

Ex: In $\mathbb{R}^3$ w/ $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, we have

$$[v]_{\mathcal{E}_3} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \rightsquigarrow \text{what w.r.t. } \mathcal{B}\,?$$

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \rightsquigarrow \begin{cases} c_1 + c_2 + c_3 = 2 \\ \quad\; c_2 + c_3 = -3 \\ \qquad\;\; c_3 = 5 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \end{array}\right] \rightsquigarrow \begin{cases} c_1 = 5 \\ c_2 = -8 \\ c_3 = 5 \end{cases}$$

$$\therefore \quad [v]_B = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underline{\begin{pmatrix} 5 \\ -8 \\ 5 \end{pmatrix}}.$$

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* Given two bases B, B' of vector spaces V and V' respectively, and given function $f: B \to B'$ there is a corresponding linear map $F: V \to V'$ with $F\left(\sum_{i=1}^{n} c_i b_i\right) = \sum_{i=1}^{n} c_i f(b_i)$.

<u>Defn</u>: A <u>change of basis matrix</u> is the matrix of a linear map $L: V \to V$ such that L is induced by a bijection $L: B \to B'$ for two bases B, B' of V.

<u>Ex</u>: Let $B' = \left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{b_1}, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{b_2}, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{b_3} \right\}$ and $B = \underline{\mathcal{E}_3} = \{e_1, e_2, e_3\}$

The change of basis matrix for these bases is...

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\therefore$ the change of basis matrix B to B' is

$$\mathrm{Rep}_{B,B'}(id) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

<u>Point</u>: Representation matrix $\mathrm{Rep}_{B,B'}(id)$ when applied to $[v]_B$ outputs $[v]_{B'}$.

I.E. $\mathrm{Rep}_{B,B'}(id) \cdot [v]_B = [v]_{B'}$.

<u>NB</u>: $R_{B,B'}(id) = \left[[b_1]_{B'} \mid [b_2]_{B'} \mid \cdots \mid [b_n]_{B'} \right]$.

<u>Ex</u>: Let $B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ and $B' = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

we compute $\mathrm{Rep}_{B,B'}(id)$ as follows: $[B' \mid B] \rightsquigarrow [I_n \mid M]$

$$\left[\begin{array}{cc|cc} -1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & -1 & -2 & -1 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & -1 & -2 & -1 \\ 0 & 2 & 3 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & -1 & -2 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$$\therefore \mathrm{Rep}_{B,B'}(id) = \begin{bmatrix} -1/2 & -1/2 \\ 3/2 & 1/2 \end{bmatrix}$$

OTOH $\mathrm{Rep}_{B',B}(id)$:

$$[B \mid B'] \rightsquigarrow [I_n \mid \mathrm{Rep}_{B',B}(id)]$$

$$\left[\begin{array}{cc|cc} 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 2 & 1 & -1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

$$\therefore \quad \mathrm{Rep}_{B',B}(id) = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}. \qquad \square$$

NB: $\mathrm{Rep}_{B,B}(id) = I_n$

↳ because it fixes each basis element.

Computationally: $[B \mid B] \rightsquigarrow [I_n \mid \underline{\underline{I_n}}] \rightsquigarrow$:)

$$\mathrm{Rep}_{B',B}(id) \cdot \mathrm{Rep}_{B,B'}(id) = \mathrm{Rep}_{B,B}(id) = I_n$$

Point: $\mathrm{Rep}_{B',B}(id) = \left(\mathrm{Rep}_{B,B'}(id)\right)^{-1}$

$$\begin{tikzcd} V_{\underline{\underline{B}}} \arrow[r, "id"] \arrow[rr, bend right, "id"'] & V_{B'} \arrow[r, "id"] & V_{\underline{\underline{B}}} \end{tikzcd}$$

Prop: An $n \times n$ matrix M is a change of basis matrix if and only if M is nonsingular.

Sketch: If M is nonsingular: then M^{-1} exists. The columns of M^{-1} form a basis B for $\mathbb{R}^n$. Hence we consider the matrix representation

$$\mathrm{Rep}_{\mathcal{E}_n, B}(id) = M : \quad [M^{-1} \mid I_n] \rightsquigarrow [I_n \mid M]$$

If M is a change of basis matrix, then

$$M = \mathrm{Rep}_{B,B'}(id), \text{ so } M^{-1} = \mathrm{Rep}_{B',B}(id). \qquad \square$$

Q: How does changing basis "play with" linear maps in general?

A: Draw a picture...

$$\begin{tikzcd}
V_{\mathcal{B}} \arrow[r, "\mathrm{Rep}_{B,A}(L)"] \arrow[d, "\mathrm{Rep}_{B,B'}(\mathrm{id})"'] & W_A \arrow[d, "\mathrm{Rep}_{A,A'}(\mathrm{id})"] \\
V_{B'} \arrow[r, "\boxed{\mathrm{Rep}_{B',A'}(L)}"'] & W_{A'}
\end{tikzcd}$$

$$\mathrm{Rep}_{A,A'}(\mathrm{id}) \cdot \mathrm{Rep}_{B,A}(L) = \mathrm{Rep}_{B',A'}(L) \cdot \underbrace{\mathrm{Rep}_{B,B'}(\mathrm{id})}_{B',B}$$

$$\mathrm{Rep}_{B',A'}(L) = \underbrace{\mathrm{Rep}_{A,A'}(\mathrm{id})}_{\text{Basis chg in } W} \cdot \underbrace{\mathrm{Rep}_{B,A}(L)}_{\text{Apply linear map } L} \cdot \underbrace{\mathrm{Rep}_{B',B}(\mathrm{id})}_{\text{Basis change in } V}$$

Point: We can represent any linear map of finite-dimensional vector spaces w.r.t. our preferred bases on the domain and codomain.

Ex: Consider the linear operator on $\mathbb{R}^3$ given by

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(L) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Let $B = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

NB: I copied B incorrectly from my notes during lecture... changes are in this color

$$\mathrm{Rep}_{B,B}(L) = \mathrm{Rep}_{B,\mathcal{E}_3}(id) \cdot \mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(L) \cdot \mathrm{Rep}_{\mathcal{E}_3,B}(id)$$

$$\begin{tikzcd}
\mathbb{R}^3_{\mathcal{E}_3} \arrow[r, "\mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(L)"] \arrow[d, shift right, "\mathrm{Rep}_{\mathcal{E}_3,B}(id)"'] & \mathbb{R}^3_{\mathcal{E}_3} \arrow[d, "\mathrm{Rep}_{\mathcal{E}_3,B}(id)"] \\
\mathbb{R}^3_B \arrow[r, "\mathrm{Rep}_{B,B}(L)"'] \arrow[u, shift right, "\mathrm{Rep}_{B,\mathcal{E}_3}(id)"'] & \mathbb{R}^3_B
\end{tikzcd}$$

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Hw: Compute $\mathrm{Rep}_{B,B}(L)$ ...

$\mathrm{Rep}_{B,\mathcal{E}_3}(id)$: $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] \rightsquigarrow \mathrm{Rep}_{B,\mathcal{E}_3}(id) = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$

$\mathrm{Rep}_{\mathcal{E}_3,B}(id)$: $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & 0 & 1/6 & 1/6 & 2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right] \rightsquigarrow \mathrm{Rep}_{\mathcal{E}_3,B}(id) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 1/6 & 1/6 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

Hence we compute $\mathrm{Rep}_{B,B}(L)$ as follows:

$$\mathrm{Rep}_{B,B}(L) = \mathrm{Rep}_{\mathcal{E}_3,B}(id) \cdot \mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(L) \cdot \mathrm{Rep}_{B,\mathcal{E}_3}(id)$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 1/6 & 1/6 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & 0 \\ 5/6 & 5/6 & 1/3 \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightsquigarrow \text{Diagonal matrix!}$$

$\boxtimes$

Point: This map L has a nicer representation with respect to B than $\mathcal{E}_3$:)

The next topic (eigenvalues, eigenvectors, and matrix diagonalization) is closely related to this idea:

Linear operators may have particularly nice representations with respect to some basis other than the standard basis...